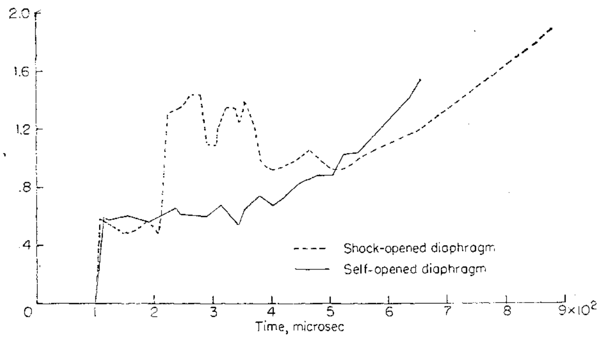

Fig. 5 Comparison of static pressure upstream of the ternary diaphragm for a shock-opened and self-opened diaphragm. Incident shock Mach number = 3.9. Static pressure port located 94.5 mm upstream of diaphragm.

mm mylar. The usable upper limit for the energy is 2.2 kjoule as greater energies break the wire. Tests showed that the cavity in the retainer ring assembly needs to be lined with a deformable, energy absorbing material to keep the wire from rebounding off of the cavity surface.

Experimental results have shown that when used a ternary diaphragm, this device has eliminated the reflected shocks observed when the diaphragm was opened by the incident shock. See Fig. 5 for a comparison of upstream static pressures. Also, flow downstream of the diaphragm is comparable to the case for which no ternary diaphragm was used, e.g., the straight expansion tube. This device may have further application as a high-speed optical shutter or as an opening device on ballistic ranges.

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Thermal Boundary-Layer Similarity at Limiting Prandtl Numbers

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Introduction

THE study of thermal boundary layers is greatly simplified by consideration of very small[1-3] or very large[3-6] Prandtl numbers. The solution is then written as a series expansion in terms of a small parameter[2,3] (the Prandtl number or its inverse, depending on the case). The principles involved in evaluating each term of the expansion are essentially the same.[2,3] For simplicity only the first term of the expansion is considered here. The case of a large Prandtl number is of particular importance since it is well known that results derived under such an assumption are still numerically adequate even for Prandtl numbers of order one (case of gases).[4]

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For instance, for $Pr \gg 1$ an incomplete gamma function[3,6] is obtained as a fundamental similar solution† when the temperature at the wall is a step function and for the most general flowfield. The problem being linear, the solution for an arbitrary wall temperature can then be written by integral superposition.[3,6] Although straightforward in principle, an integral superposition can be cumbersome to handle numerically. Furthermore since the fundamental solution has such a simple form,[3,6] it is natural to try to obtain the most general similar solution, not limited to a step function for the wall temperature.

Indeed, there exists a general class of boundary conditions at the wall (among which the earlier step function[3,6] is but a particular case) which yields a corresponding class of simple solutions. Besides its intrinsic interest the existence of a general class of similar solutions allows for the consideration of an arbitrary temperature distribution at the wall by a series expansion. If only a few terms in the series are necessary to represent the wall temperature adequately then it is advantageous to use the more explicit series solution rather than the earlier integral representation.[3,6] Furthermore the method applies equally well for temperature or heat flux prescribed at the wall, in contrast to the earlier approach[2,3,6] which can be applied in general only if the wall temperature is known.‡

Thermal Similarity

The thermal boundary layer, for both planar and axisymmetric flows is governed by,

$$u\partial T/\partial x + v\partial T/\partial y = \nu Pr^{-1}\partial^2 T/\partial y^2 \tag{1}$$

where Pr is the Prandtl number, ν the kinematic viscosity, T the temperature, u and v the components of the velocity along the wall (x direction) and normal to the wall (y direction), the fluid is incompressible and its properties constant. Call

$$\bar{x} = xU/\nu, \quad \bar{y} = yU/\nu \tag{2}$$

two dimensionless coordinates. U is an arbitrary constant with the dimensions of a velocity; for instance U could be the velocity at infinity upstream if the flow there is uniform. For $Pr \ll 1$ the thermal boundary layer is much thicker than the viscous layer and (u,v) in Eq. (1) can be replaced by the velocity at the wall of the potential flow.[1-3] For $Pr \gg 1$ the thermal boundary layer is much thinner than the viscous layer and (u,v) can be replaced by the velocity in the viscous boundary layer near the wall.[3-6] In both cases the velocity field can be represented in general by

$$u = U\beta(\bar{x})\bar{y}^m,\ v = -U[r(\bar{x})\beta(\bar{x})]'\bar{y}^{m+1}/r(m+1) \tag{3}$$

$\beta(\bar{x})$ is an arbitrary function of $\bar{x}$, $r(\bar{x})$ is equal to one for a planar flow and to the distance of a point at the wall to the axis for an axisymmetric flow. Normally m is equal to zero[2,3,6] if $Pr \ll 1$ and to one or two[3,6] if $Pr \gg 1$. The following treatment is not limited to those values of m but applies to any arbitrary m (notice that if $Pr \gg 1$ then $m \geq 1$ for the stress to be finite at the wall). It is clearly possible to construct pathological flows not included in Eq. (3), but all the flows of interest are represented if m is kept arbitrary.

When the temperature at the wall is a step function then $(T - T_\infty)/T_\infty = h(\eta)$ where T_∞ is the constant freestream temperature and η is the similarity variable which can be

† Usually,[3] similarity refers to the viscous boundary layer and imposes constraints on the outside flow. In the present context on the contrary, similarity involves only the thermal boundary layer and the outside flow remains arbitrary.

‡ In numerous problems of practical importance the temperature at the wall is unknown a priori, while some knowledge of the heat flux is available. This is the case, for instance, whenever solar radiation is the primary heating source of the wall (which could be anything from a low speed aircraft to the leaves of a plant).

defined by[3,6]

$$\eta = \tilde{y}(r\beta)^{1/(m+1)} \left[\int_{\tilde{x}_1}^{\tilde{x}} r^{\,(m+2)/(m+1)} \beta^{\,1/(m+1)}\, dx \right]^{-1/(m+2)} \tag{4}$$

$\tilde{x} = \tilde{x}_1 \geq 0$ defines the starting point of the thermal boundary layer, taking the origin of the viscous layer at $\tilde{x} = 0$. Similar solutions when the wall temperature is not constant must be of the form

$$(T - T_\infty)/T_\infty = h(\eta)\, l(\tilde{x}) \tag{5}$$

where h and l are two unknown functions and η is still defined by Eq. (4) as demonstrated below. It might be thought that Eq. (4) is unduly restrictive and that a more general $\eta(\tilde{x},\tilde{y})$ would yield a richer class of similar solutions. Any $\eta(\tilde{x},\tilde{y})$ must be a well-behaved function of $\tilde{y}$ which can be expanded for small $\tilde{y}$ or,

$$\eta = \sum_{i=1}^{\infty} \tilde{y}^{n_i}\, g_i(\tilde{x}),$$

(n_i) being a sequence of positive increasing numbers (not necessarily integers). It is always possible to define the wall position by $\eta = 0$ hence $n_1 \neq 0$ since $\tilde{y} = 0$ at the wall. One can assume further that $n_1 = 1$ (taking if necessary η^{1/n_1} for new similarity variable). Consequently at least if $\tilde{y}$ is sufficiently small, any $\eta(\tilde{x},\, \tilde{y})$ reduces to $\tilde{y}g_1(\tilde{x})$ which together with Eqs. (1) and (5) yields

$$\beta\tilde{y}^{m+1}lh'g_1' + hl'\tilde{y}^m\beta - (r\beta)'\tilde{y}^{m+1}h'g_1l/r(m+1) = h''lg_1^2/Pr \tag{6}$$

It is clear that necessary conditions for similarity are that $\beta g_1^{m-2}l'/l$ and $g^{-m-3}[\beta g' - g(r\beta)'/r(m+1)]$ be constant. Or, up to irrelevant multiplicative constants,

$$g_1 = (r\beta)^{1/(m+1)} \left[\int_{\tilde{x}_1}^{\tilde{x}} r^{\,(m+2)/(m+1)} \beta^{\,1/(m+1)}\, dx \right]^{1-/(m+2)} \tag{7}$$

$$l = \left[\int_{\tilde{x}_1}^{\tilde{x}} r^{\,(m+2)/(m+1)} \beta^{\,1/(+)}\, dx \right]^{A} \tag{8}$$

A is an arbitrary constant, $A \geq 0$ since l must be finite ($A = 0$ corresponds to a step function at the wall). Whatever $\eta(\tilde{x},\tilde{y})$ is, $l(\tilde{x})$ cannot be more general than the form given by Eq. (8), consequently there cannot be any loss in generality by taking $g_i = 0$ for $i > 1$. Then η reduces to $\tilde{y}g_1$ which is given by Eq. (4) and h satisfies the equation

$$-A(m+2)\eta^m h + \eta^{m+1}h' + (m+2)h''/Pr = 0 \tag{9}$$

Similar solutions exist if it is possible to find a constant A so that the wall temperature be of the form $h(0)l(x)$ where $l(x)$ is given by Eq. (8). To find $h(\eta)$, the general integral representation[3,6] can be used in which the wall temperature has been replaced by $h(0)l(x)$ as given by Eq. (8). Integration can then be carried out to yield an expression very similar to that which is found when the wall temperature is a step function.[3,6] It is equivalent but actually much simpler to solve Eq. (9) directly by standard methods.[7] Using as new variable

$$z = Pr\eta^{m+2}/(m+2) \tag{10}$$

Eq. (9) becomes

$$(m+2)\, zd^2h/dz^2 + (z + m + 1)\, dh/dz - Ah = 0 \tag{11}$$

and defining a new function ϕ by

$$h = \int_C \phi(p)\, e^{pz}dp \tag{12}$$

ϕ and the contour C must satisfy[7]

$$\{e^{pz}\, \phi(p)[(m+2)p^2 + p]\}_C = 0 \tag{13}$$

$$d\{\phi(p)[(m+2)p^2 + p]\}/dp - [(m+1)p - A]\,\phi(p) = 0 \tag{14}$$

The latter has the solution

$$\phi = \delta[p + 1/(m+2)]^{A-1/(m+2)}p^{-1-A} \tag{15}$$

where δ is a constant. There are basically two contours satisfying Eq. (13) each leading to a different asymptotic behavior as $z \to \infty$. The asymptotic solutions of Eq. (11) are obviously z^A and $e^{-z(m+2)}z^{-A-(m+1)/(m+2)}$. Only the latter is physically acceptable since $A \geq 0$ and h must approach zero as $z \to \infty$. A simple choice for C is obviously the interval $[-\infty,\, -1/(m+2)]$, hence

$$h = \delta \int_{-\infty}^{-1/(m+2)} \left[p + \frac{1}{m+2} \right]^{A-1/(m+2)} p^{-1-A}e^{pz}dp \tag{16}$$

which is a confluent hypergeometric function.§ Let h_0 and h'_0 be the values of h and $dh/d\eta$ at the wall. If the wall temperature is known, so is h_0. If the temperature gradient normal to the wall (heat flux) is known so is h_0' since

$$\nu(\partial T/\partial y)_{y=0} = UT_\infty h'_0 g_1(\tilde{x})l(\tilde{x}) \tag{17}$$

Eq. (16) yields at once,

$$h_0 = -\delta[-(m+2)]^{1/(m+2)}B[A + (m+1)/(m+2);\; 1/(m+2)] \tag{18}$$

$$h'_0 = \delta[-1]^m Pr^{1/(m+2)}(m+2)^{(m+1)/(m+2)}\Gamma[(m+1)/(m+2)] \tag{19}$$

and by elimination of δ

$$-h_0' = h_0 Pr^{1/(m+2)}(m+2)^{m/(m+2)}\Gamma[(m+1)/(m+2)]/B[A + (m+1)/(m+2);\, 1/(m+2)] \tag{20}$$

$\Gamma(p)$ and $B(p;q)$ are the usual gamma and beta functions. Equation (18) or (19) gives δ when the temperature or the heat flux is known at the wall. Equation (20) gives the heat flux when the wall temperature is known (and vice versa).

Conclusion

In practice it is always possible to expand the boundary condition at the wall in a Taylor-like series of the form

$$\sum_i \alpha_i \left[\int_{\tilde{x}_1}^{\tilde{x}} r^{(m+2)/(m+1)}\beta^{1/(m+1)}dx \right]^{A_i}$$

as long as it is smooth enough. The method is clearly applicable whether the temperature or the heat flux is prescribed at the wall. In the first case the temperature itself is expanded, yielding coefficients h_{oi} from Eq. (5) and δ_i from Eq. (18). In the second case $(\partial T/\partial \tilde{y})_{\tilde{y}} = {}_0/g_1(\tilde{x})$ is expanded yielding h'_{oi} from Eq. (17) and δ_i from Eq. (19). As noticed earlier if the heat flux is prescribed at the wall no other general method is available but the present one.¶ Finally a series

§ More precisely h can be reduced to a Whittaker function as Eq. (16) can be rewritten,

$$h = -\delta(-1)^m e^{-z/(m+2)} \int_0^{\infty} e^{-pz}p^{A-1/(m+2)} \times \left[p + \frac{1}{m+2} \right]^{-1-A} dp$$

Following Erdélyi's[8] notations h becomes

$$h = -\delta(-1)^m e^{-z/2(m+2)}\, \Gamma[A + (m+1)/(m+2)] \times (m+2)^{(m+3)/2(m+2)}\, z^{-(m+1)/2(m+2)} \times W[z/(m+2)]_{-A-(m+1)/2(m+2);\; -1/2(m+2)}$$

The reason for the failure of the integral representation method is quite simple. The parameter $\tilde{x}_1$ must remain variable in the fundamental solution,[3,6] since $\tilde{x}_1$ determines the position of the step function which, by integration over $\tilde{x}_1$, leads to the most general boundary condition. On the other hand the temperature gradient divided by g_1 rather than the temperature gradient itself, is the useful boundary condition at the wall. Consequently $\tilde{x}_1$ enters the general boundary condition explicitly through the definition of g_1 [Eq. (7)] and cannot remain a free parameter anymore (it must be the actual physical point where the wall is first heated). This requirement prevents the use of the fundamental solution for integral superposition when the temperature gradient is known. Notice that when the temperature is given, g_1 does not enter the boundary condition and x_1 can remain free, hence the success of the integral superposition[3,6] in that case.

representation is always simpler and more explicit than an integral representation if only a few terms are sufficient to represent the boundary condition with a good approximation.

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Effect of Uncooled Leading Edge on Cooled-Wall Hypersonic Flat-Plate Boundary-Layer Transition

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Nomenclature

b = leading-edge bluntness diam, in.
M = Mach number
Pr = Prandtl number
Re = Reynolds number
T = temperature, °R
u = velocity parallel to model, fps
x = distance from leading-edge tip, parallel to model, in.
x' = distance from leading-edge-model joint, parallel to model, in.
y = distance from plate surface, normal to surface, in.
γ = ratio of air specific heats
δ = boundary-layer thickness (point at which $\rho u/\rho_\infty u_\infty$ = 0.99), in.
ρ = density of air, lbf sec²/ft⁴

Subscripts

aw = adiabatic wall condition, assuming recovery factor of 0.85
j = value at joint between leading edge and model plate section
LE = condition of leading-edge section
o = wind-tunnel stagnation condition
t = value at transition point (defined in text)
w = condition of model plate section
∞ = wind-tunnel freestream conditions

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Hypersonic, flat-plate boundary-layer transition experiments were performed in the Arnold Engineering Development Center's von Kármán Facility 12-in. × 12-in. hypersonic wind tunnel at Mach numbers 6, 7, and 8. The flat-plate models were designed to allow investigation not only of wall cooling effects but also the effect of varying small leading-edge bluntness. The models were made up of a common flat-plate section with internal coolant flow passages and interchangeable, uncooled leading-edge sections, each with a different hemicylindrical bluntness radius and of 2-in. length.

Upon completion of the tests, examination of the data showed very little effect of wall cooling on the transition Reynolds number (defined as the product of unit Reynolds number and distance along the plate to the point of maximum pressure indicated by a Pitot tube drawn along the model surface) compared to that observed by other experimenters. As shown in Fig. 1 for a Mach number of 8, the data of Richards and Stollery[1] show a change in transition Reynolds number for a change in T_w/T_{aw} from 1 to 0.9 which is accomplished in the present experiments only by cooling to about T_w/T_{aw} = 0.6. Granted there are many factors influencing transition Reynolds number, some of which are active in causing the difference in absolute level of transition Reynolds number between the present test and those of Ref. 1 (see Ref. 2 for the influence of radiated aerodynamic noise); still, the question arose concerning a possible role the uncooled leading edge might have in reducing the influence of downstream cooling, particularly since the transition location method used in the present tests required model exposure times on the order of five minutes.

Since the leading-edge temperature was not measured, an estimate of its value at the different test Mach numbers was made by assuming it equal to the temperature the plate section attained when run without cooling. This assumption leads to estimates of leading-edge temperature ratio, T_{LE}/T_{aw}, of 0.9, 0.8, and 0.7 for M_∞ = 6, 7, and 8, respectively, for all tests regardless of the temperature subsequently reached by the cooled plate section.

Digital computer calculations were later made of the characteristics of the laminar boundary layer for a Mach number of 6, unit Reynolds number of 1.1 × 10⁶/in., assuming a perfect gas with ratio of specific heats 1.4 and Prandtl number 0.71. These calculations were made by C. H. Lewis[3] of the von Kármán Facility, using the method of Ref. 4, for the following conditions: leading edge and plate at the same temperature for uniform wall temperatures of 0.8 and 0.2 times the freestream stagnation temperature and with the leading edge at 0.8 and the remainder of the plate at 0.2 times the freestream stagnation temperature. The resulting velocity-density product profiles are compared in Fig. 2 for a station

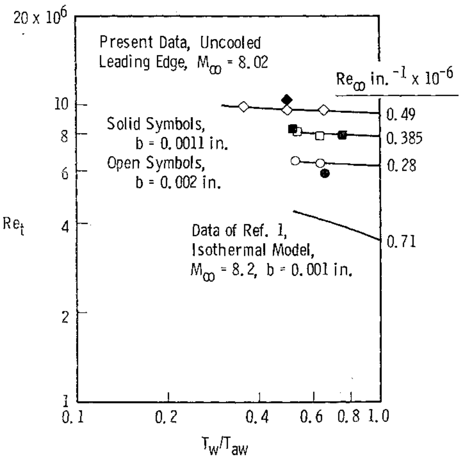

Fig. 1 Comparison of present data with isothermal model data.